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Connes' distance function on one-dimensional lattices

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Abstract

We show that there is an operator with a simple geometric significance which yields the ordinary geometry of a linear equidistant lattice via Connes' distance function.

According to Connes (see [1] and references therein) the geodesic distance function

$$d(p,q) = \text{infimum of length of paths from } p \text{ to } q \tag{1}$$

on a Riemannian manifold M can be reformulated as

$$d(p,q) := \sup\{|f(p) - f(q)| \; ; \; f \in \mathcal{A}, \; \|[\hat{\mathcal{D}}, \hat{f}]\| \leq 1\} \tag{2}$$

where $\mathcal{A}$ is a (suitably restricted) algebra of functions on M represented as multiplication operators $\hat{f}$ on a Hilbert space $\mathcal{H}$ and $\hat{\mathcal{D}}$ is the Dirac operator. The latter formulation can also be applied to discrete spaces and even generalized to 'noncommutatice spaces'. A suitable replacement for the operator $\hat{\mathcal{D}}$ has to be found, however.

In [2, 3] a one-dimensional lattice has been considered with the choice

$$(\hat{\mathcal{D}}_{s.d.}\Psi)_k = \frac{1}{2i}\,(\Psi_{k+1} - \Psi_{k-1}) \qquad k \in \mathbb{Z}\,. \tag{3}$$

The distances calculated with this *symmetric difference* operator turned out to be given by

$$d(0, 2n-1) = 2\,n\,, \qquad d(0,2n) = 2\,\sqrt{n(n+1)} \qquad (n \in \mathbb{N}) \tag{4}$$

which looks quite remote from the expected result for a linear equidistant lattice.

In the following we show that there is another operator which actually produces the expected result. For simplicity, we consider a *finite* set of N points. Then $\mathcal{A}$ is the algebra of all complex functions on it.

$f \in \mathcal{A}$ will be represented by

$$f \quad \mapsto \quad \hat{f} = \begin{pmatrix} f_1 & 0 & & & & 0 \\ 0 & \ddots & & & & \\ & & f_N & & & \\ & & & f_1 & & \\ & & & & \ddots & \\ 0 & & & & & f_N \end{pmatrix} \tag{5}$$

where $f_k := f(k)$ (numbering the lattice sites by $1, \ldots, N$). We choose the operator

$$\hat{\mathcal{D}}_N := \begin{pmatrix} 0 & \mathcal{D}_N^\dagger \\ \mathcal{D}_N & 0 \end{pmatrix} \tag{6}$$

on $\mathcal{H} = \mathbb{C}^{2N}$ where

$$\mathcal{D}_N = \begin{pmatrix} 0 & 1 & 0 & \cdots 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} . \tag{7}$$

Then $(\mathcal{A}, \mathcal{H}, \hat{\mathcal{D}}_N)$ is a *spectral triple*, a basic structure in Connes' approach to noncommutative geometry [4] (see also [5] for a refinement). It is called *even* when there is a grading operator. In the case under consideration such an operator is given by

$$\gamma := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} . \tag{8}$$

It is selfadjoint and satisfies

$$\gamma^2 = \mathbf{1} , \qquad \gamma \, \hat{\mathcal{D}}_N = -\hat{\mathcal{D}}_N \, \gamma \qquad \gamma \, \hat{f} = \hat{f} \, \gamma . \tag{9}$$

Let us now turn to the calculation of the distance function. With a complex function f we associate a real function F via

$$F_1 := 0 , \quad F_{k+1} := F_k + |f_{k+1} - f_k| \qquad k = 1, \ldots, N-1 . \tag{10}$$

Then $|F_{k+1} - F_k| = |f_{k+1} - f_k|$ and

$$\| [\hat{\mathcal{D}}_N, \hat{f}] \, \psi \| = \| [\hat{\mathcal{D}}_N, \hat{F}] \, \psi \| \tag{11}$$

for all $\psi \in \mathbb{C}^{2N}$. Consequently, in calculating the supremum over all functions f in the definition of Connes' distance function, it is sufficient to consider only *real* functions. Then $Q_N := i \, [\hat{\mathcal{D}}_N, \hat{f}]$ is hermitean and its norm is given by the maximal absolute value of its eigenvalues. Instead of Q_N it is simpler to consider

$$Q_N \, Q_N^\dagger = \mathrm{diag} \left(0, (f_2 - f_1)^2, \ldots, (f_N - f_{N-1})^2, (f_2 - f_1)^2, \ldots, (f_N - f_{N-1})^2, 0 \right) \tag{12}$$

which is already diagonal. This implies

$$\| [\hat{\mathcal{D}}_N, \hat{f}] \| = \max \{ |f_2 - f_1|, \ldots, |f_N - f_{N-1}| \} \tag{13}$$

from which we conclude that $d(k, l) = |k - l|$.

The choice (3) for $\hat{\mathcal{D}}$ was motivated by a simple discretization procedure (which is known to cause the problem of fermion doubling in lattice field theories). There is, however, no reason why this operator must yield the plain geometry of a linear equidistant lattice via Connes' distance function. There are

many geometries which can be assigned to a discrete set and these should correspond to the choice of some operator $\hat{\mathcal{D}}$. Now it is certainly of interest to know what distinguishes our choice (6). This is built from the operator $\mathcal{D}$ in such a way that $\hat{\mathcal{D}}$ is selfadjoint. Moreover, the construction guarantees that there is a grading operator. So we are left to understand the significance of $\mathcal{D}$. This matrix is the adjacency matrix of the oriented linear lattice graph (see Fig. 1).

Fig. 1

An oriented linear lattice graph.

This digraph plays a basic role in a formulation of lattice theories in the framework of noncommutative geometry [6] (see also the references cited there).

Remark. Instead of using $\hat{\mathcal{D}}$ to define Connes' distance function, we may use directly $\mathcal{D}$ (which, in general, is not symmetric) and no doubling in the representation of f. A simple calculation in the case treated above shows that

$$\|[\mathcal{D}_N, f]\| = \|[\hat{\mathcal{D}}_N, \hat{f}]\| \tag{14}$$

so that we obtain the same distances as before. ■

Let us now turn to a *closed* linear lattice. Connecting in addition the last with the first point in the oriented digraph in Fig. 1, the adjacency matrix becomes

$$\mathcal{D}_{Nc} = \begin{pmatrix} 0 & 1 & 0 & \cdots 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} . \tag{15}$$

For $\Psi = (\phi, \psi) \in \mathbb{C}^{2N}$ we find

$$\begin{aligned} \|[\hat{\mathcal{D}}_{Nc}, \hat{f}]\,\Psi\|^2 &= \|[\mathcal{D}_{Nc}, f]\,\psi\|^2 + \|[\mathcal{D}^\dagger_{Nc}, f]\,\phi\|^2 \\ &= \sum_{k=1}^{N} |f_{k+1} - f_k|^2\,(|\psi_{k+1}|^2 + |\phi_k|^2) \\ &\geq \sum_{k=1}^{N} |\,|f_{k+1} - a| - |f_k - a|\,|^2\,(|\psi_{k+1}|^2 + |\phi_k|^2) = \|[\hat{\mathcal{D}}_{Nc}, \hat{F}]\,\Psi\|^2 \end{aligned} \tag{16}$$

where $F_k = |f_k - a|$. Here and in the following, an index $N+1$ has to be replaced by 1. Choosing $a = f_1$, we have $\|[\hat{\mathcal{D}}_{Nc}, \hat{F}]\| \leq \|[\hat{\mathcal{D}}_{Nc}, \hat{f}]\|$ and $F_k = |f_k - f_1|$. It follows that

$$d(1,n) = \sup\{|F_n|\,;\; F \text{ real},\, F_1 = 0,\, \|[\hat{\mathcal{D}}_{Nc}, \hat{F}]\| \leq 1\} \tag{17}$$

The condition $\|[\hat{\mathcal{D}}_{Nc}, \hat{F}]\| \leq 1$ is equivalent to $|F_{k+1} - F_k| \leq 1$, $k = 1, \ldots, N$. Let $n - 1 \leq N - n + 1$. It is then possible to set the first $n-1$ terms in the identity

$$\underbrace{(F_2 - F_1) + \cdots + (F_n - F_{n-1})}_{n-1 \text{ terms}} + \underbrace{(F_{n+1} - F_n) + \cdots + (F_1 - F_N)}_{N-n+1 \text{ terms}} = 0 \tag{18}$$

each separately to 1. Using the trivial identity

$$|F_n| = |(F_2 - F_1) + (F_3 - F_2) + \cdots + (F_n - F_{n-1})| \tag{19}$$

we now find $d(1,n) = n - 1$. If $n - 1 > N - n + 1$, then each of the last $N - n + 1$ terms in (18) can be set to 1. Using

$$|F_n| = |(F_1 - F_N) + (F_N - F_{N-1}) + \cdots + (F_{n+1} - F_n)| \tag{20}$$

we find $d(1,n) = N - n + 1$.

In Connes' noncommutative geometry the commutator $[\hat{\mathcal{D}}, \hat{f}]$ represents a 'differential' $\mathrm{d}f$. The inequality which appears in the definition of the distance function can then be written as $\|\mathrm{d}f\| \leq 1$. Given a differential calculus (in the abstract algebraic sense), in order to have a distance function we need a definition of the norm of $\mathrm{d}f$. Connes defines it via a representation of the (first order) differential algebra. In the case of discrete sets, it is natural to define a norm by

$$\|\mathrm{d}f\| = \sup\{|f(k) - f(l)|/\epsilon_{kl}\,;\ (kl) \in E\} \tag{21}$$

where a (di)graph structure has been assigned to the set by the first order differential calculus (see [6] for details) and E denotes the set of its arrows. The constants ϵ_{kl} are introduced here for convenience if one is interested in a continuum limit. The distance function is then taken to be

$$d(p,q) := \sup\{|f(p) - f(q)|\ ;\ f \in \mathcal{A},\ \|\mathrm{d}f\| \leq 1\}\ . \tag{22}$$

This receipe now reproduces the ordinary distances on the underlying graph.[1]

Further results on distance functions à la Connes on finite sets will be published elsewhere [7].

References

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[1] The distance evidently does not depend on the orientation of arrows in the digraph.